# Computer driven theorems and questions in geometry

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1. **Preliminaries:** Given an orientable surface $S$ with negative Euler characteristic, a minimal set of generators of the fundamental group of $S$, and a hyperbolic metric on $S$, each free homotopy class $C$ of closed oriented curves on $S$, determines three numbers: the word length (that is, the minimal number of generators and inverses necessary to express $C$ as a cyclically reduced word), the minimal self-intersection and the geometric length.

   We will discuss how to compute these three numbers. In particular, we will talk about parametrizations of the Techmuller space of the pair of pants and the torus with one boundary component and algorithms to compute the self-intersection of curves on surfaces [2], [15], [5].

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2. **Statistics of curves on surfaces** In a geometric vein, Lalley [18] determined the mean of the self-intersection of all closed geodesic up to a given length on a hyperbolic closed surface .

   Pollicot and Sharp [24] gave asymptotic results for self-intersections of closed geodesics on surfaces for which the angle of the intersection occurs in a given arc, Pollicot [23] gave asymptotic estimates for pairs of closed geodesics, the differences of whose lengths lie in a prescribed family of shrinking intervals.

   In a topological vein: Chas and Lalley [11] showed that the distribution of self-intersections of closed curves on a surface, sampling by word-length.(appropriately normalized) converges to a Gaussian when the word length tends to infinity. Wroten [30] extended this result to closed surfaces.

   Chas, Li, and Maskit [12] experimentally studied the distribution of geometric length sampling by word length.

3. **Intersection number of curves and their relation to word/geometric length:** In order to have "large" self-intersection, a free homotopy class of curves $C$ has to be "long" in terms of word and geometric length. How long it must be is studied in [1] (for hyperbolic surfaces and geometric length), [13] and [14] (for the punctured torus and pairs of pants respectively, in terms of the word-length).

4. **Growth rates of curves on a surface:** The number of closed geodesics of length less than $L$ on a closed surface grows asymptotically like $\frac{e^L}{L}$. Thus, being independent of the topology of the surface, the growth rate of the closed geodesics merely reflects a property of the hyperbolic plane. On the other hand, in the last decade Mirzakhani [22] answered the long standing question of the growth of *simple* closed geodesics, by showing they grow asymptotically like a polynomial whose degree is the dimension of the Teichmüller space of the surface, and whose leading constant depends on the hyperbolic surface. In a related vein to growth rates of simple geodesics Mirzhakani computed the Weil-Peterson volume of Moduli space [21]. Rivin showed that the same growth rate of simple geodesics occurs for geodesics with one self-intersection [26]. The case of more than one self-intersection is still open, although there are partial results on the pair of pants by Mirzakhani's student Jenya Sapir [27].

5. **Relation with algebraic structures:** In the eighties, Goldman [16] discovered an unexpected Lie algebra structure on linear combinations of free homotopy classes of directed closed curves

on an orientable surface. This Lie algebra is defined by combining two well known operations on curves: the transversal intersection and the composition of directed loops which start and end at the same point. Turaev [29] found a Lie coalgebra in the same space, that satifies a compatibility equation with Goldman's. The Lie bialgebra turns out to be a powerful tool and its structure still contains many mysteries. Chas [6], Chas and Krongold [9, 10], Chas and Gadgil [8], Cahn [3] and Cahn and Chernov [4] studied how this Lie bracket and its generalizations relate to the intersection of curves on surfaces. LeDonne [19] completed Chas work by studing the relation of Turaev's cobracket and the intersection.

6. **Equivalent curves on surfaces** Two free homotopy classes of closed curves are said to be *length equivalent* if for any hyperbolic structure on the surface, the length of the geodesic in one class is equal to the length of the geodesic in the other class. By [17] and [25] we know that these length equivance classes, although finite, have unbounded cardinality.

   Leininger [20] studied length-equivalence classes, and related this study to other equivalence relations of the same type . Chas [7] found examples of length equivalent curves with different self-intersection. Dylan Thurston [28] proved a new and somewhat striking result about intersection of multicurves. Jonah Gaster use D. Thurston's ideas to give a new characterization of classes of equivalent curves on surfaces.

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