Mini course: Identities on hyperbolic surfaces

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This minicourse will focus on the study of geodesics on hyperbolic surfaces and how various geometric invariants on a hyperbolic surface interact.

The first two lectures will essentially be a crash course on hyperbolic surfaces (that is, Fuchsian groups). After that the main focus of the rest of the lectures will be on hyperbolic surface identities. An *identity on a surface* is an equation which relates an infinite sum with a geometric or topological invariant of the surface. The index set of the sum will be a particular set of geodesics and the terms of the sum depend only on the length of the geodesic being summed over. The equations are independent of the choice of hyperbolic structure on the underlying surface thus justifying the name identity. For example, the following two identities are sums over the same index set which is the set of geodesic paths orthogonal to ∂S in both endpoints (called orthogeodesics). They appear in the papers [1] and [5], respectively.

Theorem 0.1. *Let S be a hyperbolic surface with non-empty boundary ∂S. Then*

$$\ell(\partial S) = \sum_{h \in L_S} 2 \log \left(\coth \frac{\ell(h)}{2} \right) \quad \textit{([1])}$$

and

$$\mathrm{Vol}(T^1(S)) = \sum_{h \in L_S} 4 \mathcal{R} \left(\left(\mathrm{sech} \frac{\ell(h)}{2} \right)^2 \right) \quad \textit{([5])}$$

where L_S is the set of orthogeodesics of S and $\mathcal{R}$ is Rogers' dilogorithm function.

- **Lecture 1:** Crash course on Fuchsian groups I. The hyperbolic plane, algebraic and geometric classification of isometries, discrete groups and properly discontinuous actions, and necessary conditions for discreteness. Equivalent descriptions of a hyperbolic surface: as the quotient of a torsion-free Fuchsian group, as an oriented Riemannian manifold of constant curvature minus one, and as (a complex analytic object) a Riemann surface. Closed geodesics on a hyperbolic surface. The limit set and the set of discontinuity of a Fuchsian group.

- **Lecture 2:** Crash course in Fuchsian groups II.

  Hausdorff dimension of the limit set and its one dimensional measure. Classification of elementary groups. Non-elementary groups and some of their properties. Hyperbolic geometry: right angled hexagons, pairs of pants, and their moduli (Fenchel-Nielsen coordinates). Pants decomposition of a surface. The collar lemma. Teichmüller space and moduli space.

- **Lecture 3:** Identities on hyperbolic surfaces. The orthospectrum and the two identities mentioned above. Birman-Series result that simple geodesics on a hyperbolic surface have Hausdorff dimension one. Identities for cusped (no boundary) finite area surfaces ([7]).

- **Lecture 4:** Identities for closed surfaces and other identities, growth rate of closed geodesics and Mirzakhani's result on growth rate of simple closed geodesics.

Some references follow.

References

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[4] B.H. Bowditch. A proof of McShane's identity via Markoff triples. Bull. London Math. Soc., 28 (1996), 73-78. MR 1356829 284 MR1356829 (96i:58137)

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[8] McShane, Greg, On the variation of a series on Teichmller space. Pacific J. Math. 231 (2007), no. 2, 461-479. (Reviewer: Martin Möller) 32G15 (30F60 57M50)

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